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An Interactive Code for Solving Differential Equations Using Phase Integral Methods

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The solution of ordinary differential equations by the means of Phase Integral Methods is eminently suited for an interactive computational system with real time graphics. We describe a code devised for this purpose and illustrate its use.

I. INTRODUCTION

The power and simplicity of Phase Integral Methods for the approximate solution of differential equations make them a common tool in many branches of physics, and a particularly useful one in plasma physics, where the equations are often too cumbersome to solve by standard exact methods. Many of the differential equations of interest can be put in the form

$$\frac{d^2\psi}{dz^2} + Q(z, \omega)\,\psi = 0. \tag{1}$$

In these cases, the existence of solutions and the approximate complex eigenfrequencies ω can often be determined by phase integral methods. In this work we show that phase integral solutions of these equations are efficiently obtained by interactive computation, a mode of analysis particularly suited for these problems since pattern recognition and guesswork are needed as well as complicated calculations.

In Section II we briefly review the Method of Phase Integrals as it applies to the restricted problem of determining whether solutions of Eq. (1) exist, and if so, finding the associated eigenfrequencies.

Section III consists of a general description of the code developed for this purpose along with examples of its use.

II. THE METHOD OF PHASE INTEGRALS

A. *Stokes Structure*

We take Eq. (1) as standard form for the differential equation to be examined. The complex frequency, ω, generally plays the role of an unknown eigenvalue, and

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$z = x + iy$ is a complex variable. The physical problem is initially defined on the real axis and the equation has been analytically continued into the complex plane.

Briefly, the WKBJ approximate solutions of Eq. (1) (so named after Wentzel, Kramers, Brillouin, and Jeffreys) [2] take the form

$$\psi_{\pm} = Q^{-1/4} \exp\left(\pm i \int^{z} Q^{1/2}\, dz\right), \tag{2}$$

and provided that

$$\frac{dQ}{dz} Q^{-3/2} \ll 1 \tag{3}$$

a general solution of Eq. (1) can be approximated by

$$\psi = a_{+}\psi_{+} + a_{-}\psi_{-}\,. \tag{4}$$

The solutions $\psi_{\pm}$ are local, not global solutions of Eq. (1). Clearly, inequality (3) is not valid in the vicinity of a zero of $Q(z, \omega)$, commonly called a turning point. Aside from this, however, $\psi_{\pm}$ are not approximations of a continuous solution of Eq. (1) in the whole z plane; i.e., if ψ is to approximate a continuous solution of Eq. (1), then the coefficients a_{+}, a_{-} are not fixed over the whole z plane. The Method of Phase Integrals consists in relating, for a given solution of Eq. (1), the WKBJ approximation in one region of the z plane to that in another.

These regions are separated by the so-called Stokes and anti-Stokes lines associated with $Q(z, \omega)$, and thus the qualitative properties of the solution are determined once these lines are known. The Stokes (anti-Stokes) lines associated with $Q(z, \omega)$ are paths in the z plane, emanating from zeros or singularities of $Q(z, \omega)$, along which $\int Q^{1/2}(z, \omega)\, dz$ is imaginary (real). We review first the characteristic properties of these lines and then, in Section IIB, the way in which they determine the global nature of a WKBJ solution. We introduce a method of determining these lines which does not involve time consuming integration. Define a local anti-Stokes line to be, for any z_0, an infinitesimal path dz emanating from z_0 along which $Q^{1/2}\, dz$ is real. Along this path $|\psi_{\pm}|$ is essentially constant; i.e., the solutions are oscillatory. If $Q(z_0, \omega)$ is finite and well behaved, the local anti-Stokes line is given by setting dz equal to a real number times $\pm Q(z_0)^{-1/2}$; i.e., from z_0 there issue two oppositely directed lines. Points at which $Q(z_0)$ is zero or infinity must be analyzed with more care. When the zero is first order, we may in general write $Q(z) = A(z)(z - z_0)$, and determine the direction of local anti-Stokes lines from the requirement that $Q^{1/2}\, dz = A(z_0)^{1/2}\, dz^{3/2} \equiv |A(z_0)|^{1/2} \exp[(i\theta/2)]\, dz^{3/2}$ be real. Since dz is then proportional to $\exp[i(-\theta + 2n\pi)/3]$ with n integer, we find that three anti-Stokes lines emanate from z_0. Similarly, one finds that from a double root there issue four anti-Stokes lines, from a simple pole a single line, etc. It is thus quite easy to read the locations of zeros, poles, etc., of a function from a plot of the z plane upon which are displayed the local anti-Stokes lines, which we will refer to as a Stokes diagram. An example is shown in Fig. 1 for a function Q which possesses simple zeros in the first and third quadrants,

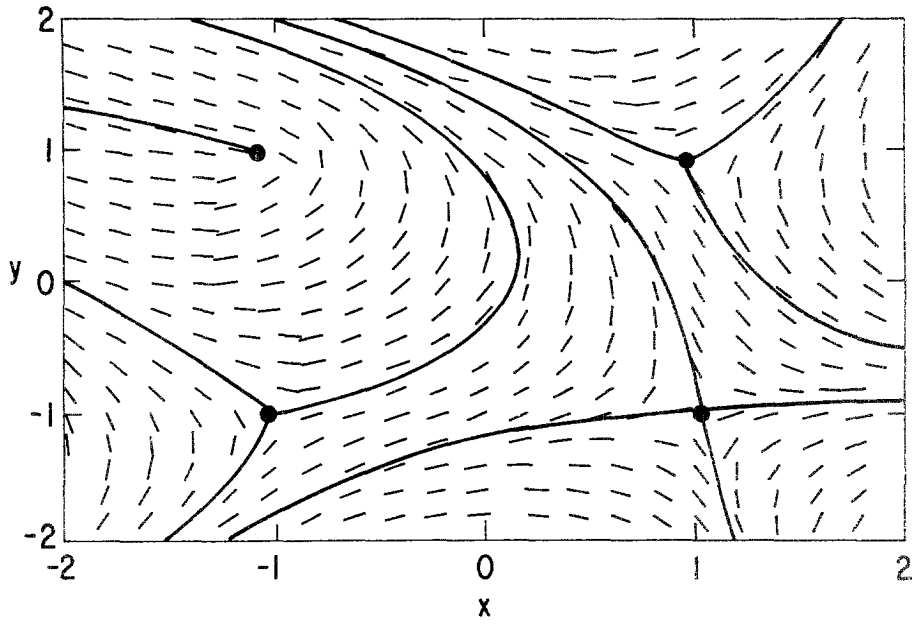

FIG. 1. Plot of the local anti-Stokes structure for $Q = (z - z_1)(z - z_2)(z - z_3)^2/(z - z_4)$ with $z_1 = 1 + i$, $z_2 = -1 - i$, $z_3 = 1 - i$, and $z_4 = -1 + i$. The global anti-Stokes lines have also been added.

a double zero in the fourth quadrant, a pole in the second quadrant, and no other zeros or singularities. In referring to Stokes diagrams, we will refer to both zeros and singularities of $Q(z)$ as singular points of the diagram since it is the function $Q^{1/2}$ which is relevant in this diagram.

A display of this nature allows a qualitative survey of the analytic structure of a function, without the numerical complication of an actual search for roots. It is also more convenient than an integral method utilizing closed contours such as that due to Nyquist [2].

Using the local anti-Stokes lines as guides, we can form global, continuous anti-Stokes lines for those particular lines which emerge from the singular points of the Stokes plot, and these lines have been added to Fig. 1. Along the global anti-Stokes lines the functions $\psi_\pm$ are, within the validity of the WKBJ approximation, of constant amplitude, i.e., oscillatory. In performing these connections, keep in mind that away from the singular points, the lines cannot meet or cross. We similarly define local and global Stokes lines to be lines for which the integral $\int Q^{1/2}\, dz$ is imaginary. Along the Stokes lines the WKBJ solutions are exponentially increasing or decreasing with fixed phase. Except at singular points the Stokes and anti-Stokes lines are orthogonal. The global anti-Stokes and Stokes lines which are attached to the singular points of the Stokes diagram, along with the Riemann cut lines, describe the global properties of the WKBJ solutions.

As an illustration of a more complicated structure, in Fig. 2 are shown the local anti-Stokes lines associated with the plasma dispersion function [3] $Z(z)$, defined as the Hilbert transform of the gaussian, $Z(z) = \pi^{-1/2} \int_{-\infty}^{\infty} dt \exp(-t^2)(t - z)^{-1}$ for $\mathrm{Im}\, z > 0$ and as the analytic continuation of this for $\mathrm{Im}\, z < 0$. This function is quite complicated in the lower half plane. During operation of the code, the local anti-Stokes lines of any region can be examined using a magnification limited only by the numerical accuracy with which $Q(z, \omega)$ is computed, thus allowing the examination of local analytic properties of the differential equation. Thus, detail such as is shown in Fig. 2 is easily investigated.

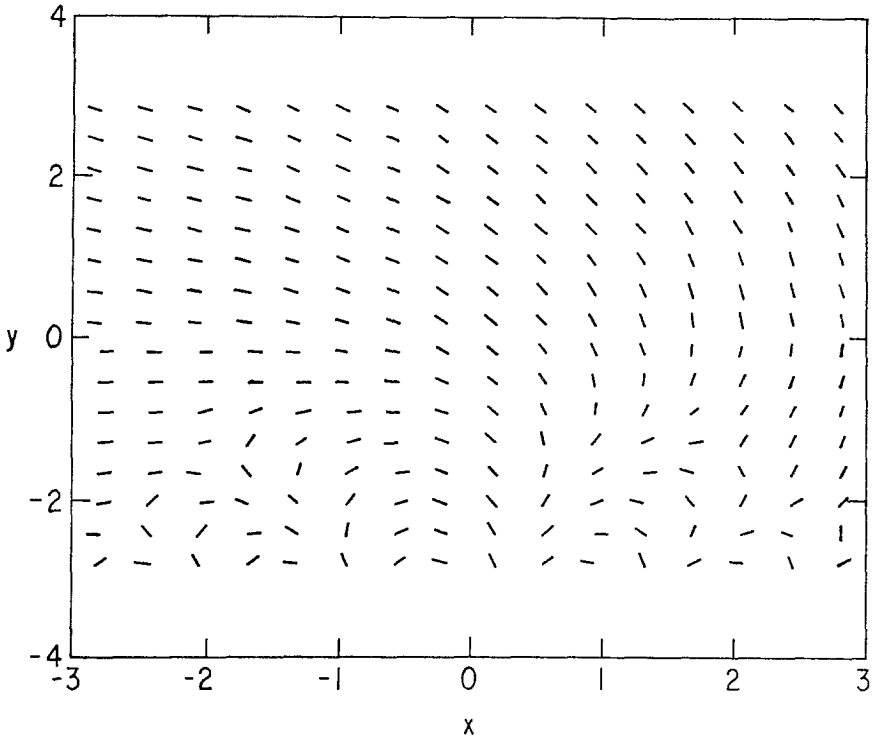

FIG. 2. The local anti-Stokes plot for the plasma dispersion function. This function exhibits great complexity in the lower half plane. The code permits a magnified view of any region of the Stokes diagram, thus allowing examination of detailed analytic structure if desired.

B. *Connection Formulas*

The canonical Stokes structure for a solution associated with a pair of real turning points is shown in Fig. 3. For illustration, we have chosen $Q = L/(z^2 + 1) - \omega$ which has turning points located at $\pm v = \pm(L/\omega - 1)^{1/2}$, which are located on the real axis provided ω is real and between zero and L. Note that for $|z|$ large $Q \to -\omega$, which means that the local anti-Stokes lines all point in the same direction; vertical for ω real and positive. In addition to the anti-Stokes lines emanating from

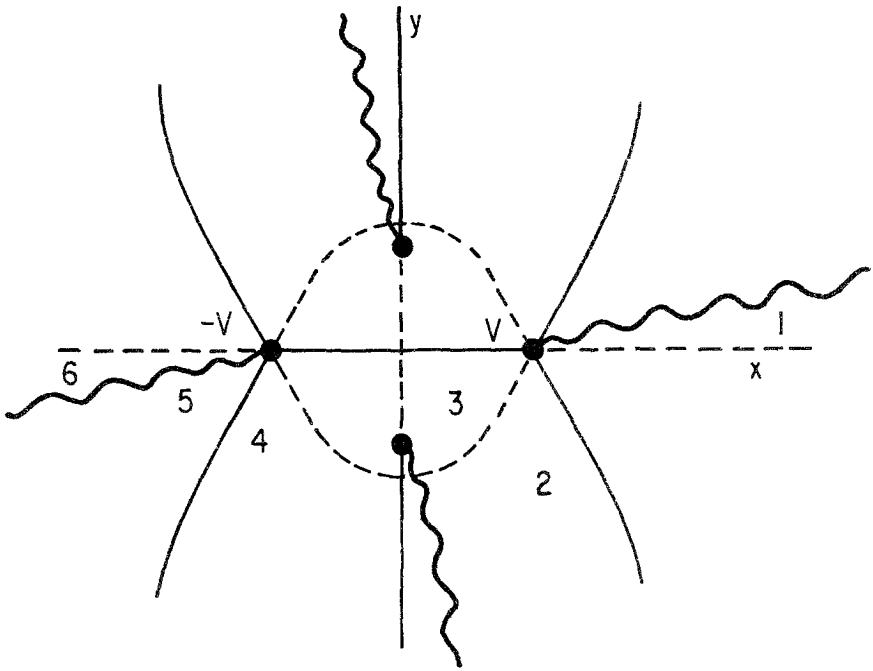

FIG. 3. The canonical Stokes structure for an equation with an attractive potential with two real turning points. Shown are the global anti-Stokes lines (solid), Stokes lines (dotted), and branch cuts (wavy).

the turning points, we have added the Stokes lines (broken lines). There are three separate regions of the real axis where the WKBJ expression can give a good approximation to a solution, and these regions are isolated from one another by the turning points.

In the notation of Heading, ignoring the slow $Q^{-1/4}$ dependence, a WKBJ solution is denoted by

$$(z, v)_s = \exp\left(i \int_v^z Q^{1/2}\, dz\right) \tag{5}$$

where the subscript $s(d)$ indicates that the solution is subdominant (dominant); i.e., exponentially decreasing (increasing) for increasing $| z - v |$ in a particular region of the z plane, bounded by Stokes and anti-Stokes lines. The point v is taken to be a nearby turning point to which the dominancy and subdominancy refers. Here increasing $| z - v |$ is made precise by specifying that it signify following the local Stokes lines in the direction that $| z - v |$ increases. The two independent local WKBJ approximate solutions of Eq. (1) in this notation are given by (z, v) and (v, z). Clearly if (z, v) is subdominant, then (v, z) is dominant. It is readily verified that upon crossing an anti-Stokes line these two solutions reverse character.

The rules given by Heading for obtaining a globally defined WKBJ solution which corresponds to the approximation of a single solution of the differential equation

are the following. Begin with a particular solution in one region of the z plane, choosing that combination of subdominant and dominant solutions which gives the desired boundary conditions in this region. The global solution is obtained by continuing this solution through the whole z plane effecting the following changes:

(1) If a_d and a_s are respectively the coefficients of the dominant and subdominant terms of a solution, then upon crossing a Stokes line in counterclockwise sense a_s must be replaced by $a_s + Ta_d$ where T is called the Stokes constant. When the Stokes line originates at a first order zero, $T = +i$.

(2) Upon crossing a cut in a counter clockwise sense, the cut originating from a first order zero of Q at the point v, we have

$$\begin{aligned}(v, z) &\to -i(z, v)\\ (z, v) &\to -i(v, z).\end{aligned} \tag{6}$$

The property of dominancy or subdominancy is preserved in this process.

(3) Upon crossing an anti-Stokes line, subdominant solutions become dominant and vice verse.

These rules describe the results of analytically continuing the WKBJ solutions around the branch points and of integrating Eq. (1) through the turning points, in the neighborhood of which the WKBJ solutions are invalid, but the function Q is dominated by its behavior near the turning point and the integration can be done directly. The connection of the WKBJ solutions can also be carried out at some distance from the turning point in the complex plane where these solutions are valid. [4] In this manner we can pass from region to region across the cuts and Stokes lines emanating from the turning points. Beginning with any combination of dominant and subdominant solutions in one region, this process leads to a globally defined single valued approximate solution of Eq. (1). Although it would appear that the first rule gives rise to a discontinuous solution, this is not the case. At the Stokes line, in the presence of a dominant solution, the discontinuity produced is small compared to the error due to the WKBJ approximation itself. As one continues further away from the Stokes line, however, the subdominant term will begin to be important, and the modified coefficient is the correct one.

We illustrate this process for the case of Fig. 3. Assuming that we wish a solution which is exponentially decreasing for $x \to \infty$, we begin by taking ψ to be the subdominant function $(z, v)_s$ in region 1 of Fig. 3. In region 2 this function becomes dominant, so that $\psi = (z, v)_d$. Passing into region 3 we find the continuation

$$\psi = (z, v)_d - i(v, z)_s . \tag{7}$$

To proceed further we attach the solution to the left hand turning point, using the identity

$$(z, c) = (z, a)(a, c). \tag{8}$$

finding also in region 3

$$\psi = (z, -v)_d(-v, v) - i(-v, z)_s(v, -v). \tag{9}$$

Note that the Stokes line connecting the two poles is not attached to the turning points of the solution and has no effect on the continuation.

To pass into region 4 we cross a Stokes line in the counter clockwise direction and thus find

$$\psi = -i[(v, -v) + (-v, v)](-v, z)_s + (-v, v)(z, -v)_d \tag{10}$$

and thus in region 5

$$\psi = -i[(v, -v) + (-v, v)](-v, z)_d + (-v, v)(z, -v)_s . \tag{11}$$

Crossing the cut into region 6 we find

$$\psi = [(v, -v) + (-v, v)](z, -v)_d + i(-v, v)(-v, z)_s . \tag{12}$$

In any case, whether we push the cut upward, letting region 5 include the real axis, or leave it as shown in Fig. 4, the connection formula remains the same. Requiring that the solution tend to zero at $x \to -\infty$, we set the coefficient of the dominant solution equal to zero, or

$$(v, -v) = -(-v, v) \tag{13}$$

which has the solution

$$\int_{-v}^{v} Q^{1/2}\, dz = [n + (1/2)]\,\pi \tag{14}$$

for n integer. This phase integral condition determines the eigenfrequency ω of Eq. (1) and is identical to the Bohr–Sommerfield condition in quantum mechanics [5].

The accuracy of the WKBJ determination of an eigenvalue is, in practice, usually much greater than an evaluation of Eq. (3) would lead one to expect. An extreme case is the harmonic oscillator potential, $Q = \omega - x^2$, for which the WKBJ determination leads to the exact expression $\omega = 2n + 1$, n integer. For many problems the potential well is approximately that of a harmonic oscillator and thus the eigenvalue is very accurately determined. Equation (3) is misleading since near the turning points, where it is certainly not satisfied, the derivation of the eigenvalue condition makes use of the exact (Airy function) solution. Also the integral, Eq. (14), is insensitive to the exact form of Q, which is in any case closely prescribed by the two turning points. However, the integral is often a sensitive function of the phase and magnitude of ω, thus leading to an accurate determination. In the case of the example of Fig. 3, an expansion about $z = 0$ leads to the approximate harmonic potential $Q_h = L - \omega - Lz^2$, giving $\omega = L[1 - (2n + 1)/L^{1/2}]$. For $n = 0$, $L = 100$ this gives $\omega = 90$. Using the complete form, $Q = L/(1 + z^2) - \omega$, the code described here converges to $\omega = 90.2$, and a direct numerical integration of the differential equation gives $\omega = 90.7$.

For the physical interpretation of the solution and the understanding of the boundary conditions, it is necessary to recall the derivation of the original equation, which is normally through Fourier Laplace transformation of a time dependent equation. The solution for fixed ω has the form $\psi(x)\exp(-i\omega t)$. In this particular case the differential equation is self adjoint, there are no sources or sinks of energy, and this is reflected in the Stokes diagram by the fact that the asymptotic x axis exactly coincides with a Stokes line; i.e., the solution is asymptotically exponentially damped in space, with no propagation in or out. In Section III we present examples in which the differential equation is not self adjoint, and a solution necessitates asymptotic propagation.

The canonical two turning point Stokes structure shown in Fig. 3 is the pattern that one must search for in a standard eigenvalue problem. In the case of an algebraically complicated function $Q(z, \omega)$, perhaps dependent on several physical parameters, the job of analyzing the Stokes structure associated with Eq. (1) for all physically interesting parameter values can be quite tedious. Further, the global properties of Stokes regions can change drastically for small changes in the eigenvalue ω. Thus a means of rapidly exploring this structure in order to ascertain the possibility of the existence of a mode, and in case such a possibility exists to iteratively find the associated frequency ω is extremely useful.

III. The Numerical Search for WKBJ Solutions

A. *General Description*

The code developed can be used to analyze any differential equation of the form of Eq. (1), and in particular to iteratively find the eigenfrequencies associated with particular solutions. It is written in Fortran, and application to a specific problem only involves inserting the lines necessary to construct the function $Q(\omega, z)$.

It is written for interactive use on a console with a screen for plotting. Upon execution of the code, an initial guess for the frequency must be entered and the resulting Stokes structure (local anti-Stokes lines) is displayed. In general, there may be present a number of singular points of various type. If no turning points likely to give a solution of the form desired are observed, a new frequency guess is made and the new Stokes structure is displayed. When a likely pair of turning points are observed, their approximate locations are entered, and the code then locates them to the desired degree of accuracy. Then the integral Eq. (14) is evaluated between these points and the result is displayed. At this point, the desired mode number can be selected, and the code then proceeds to find the correct eigenfrequency iteratively, displaying the resulting Stokes structure, the integration path, the integrand, and the path of the search in the frequency plane. For efficient use of the code, these must all be observed and made use of to arrive quickly at a solution. A more detailed description of the use of this output has been given in a short guide to the actual operation of the code. [6]

The search routine for the frequency consists of stepping the frequency, by an entered amount, four different directions in the complex plane, and for each step finding the new zeros of $Q(z, \omega)$, calculating the integral, and comparing it to the integral obtained with the original ω. The best resulting ω in terms of satisfying Eq. (14) is then selected for the next iteration. In case none of these stepped values result in a better solution of Eq. (14) than the original value, which will certainly occur when ω is very near the correct eigenvalue, the step distance is decreased and the search is repeated. When improvement of the eigenvalue is impossible, due to errors occurring from the finite number of integration points and the limited accuracy with which the zeros of $Q(z, \omega)$ are found, the frequency search routine will fail and the routine will terminate.

In some cases the turning point locations can be very sensitive functions of ω, and a large step in ω will cause them to be lost. The search for the zeros of $Q(z)$ and the integration paths are displayed on the Stokes diagram during the four-way frequency search, so that if this problem occurs it will be apparent and a smaller value for the frequency step can be selected.

During operation the plot of the integrand indicates how accurately the zeros of $Q(z, \omega)$ are being found; if it is not relatively small at the end points, the acceptable error must be reduced. A sample plot of the integrand is shown in Fig. 4. This plot as well as the Stokes plot warns of the presence of zeros or singularities near the integration path. This is relevant since the integration path must be analytically continuable to a path following the anti-Stokes lines connecting the two turning points. In Fig. 5 is shown, with a dotted line, a typical (straight line) integration path joining

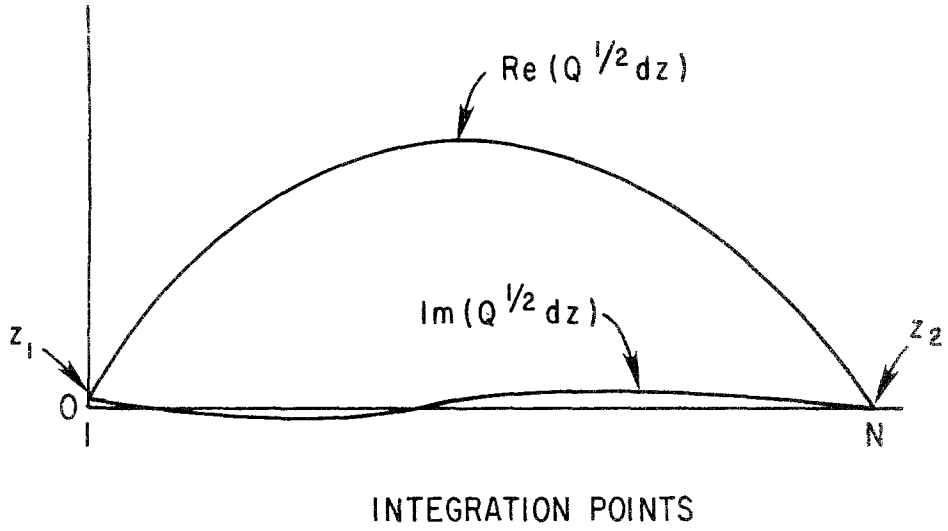

FIG. 4. A typical plot of the integrand $Q^{1/2}\,dz$ for ω nearby a solution. The integrand must be relatively small at the end points, otherwise the accuracy of the turning point location must be improved.

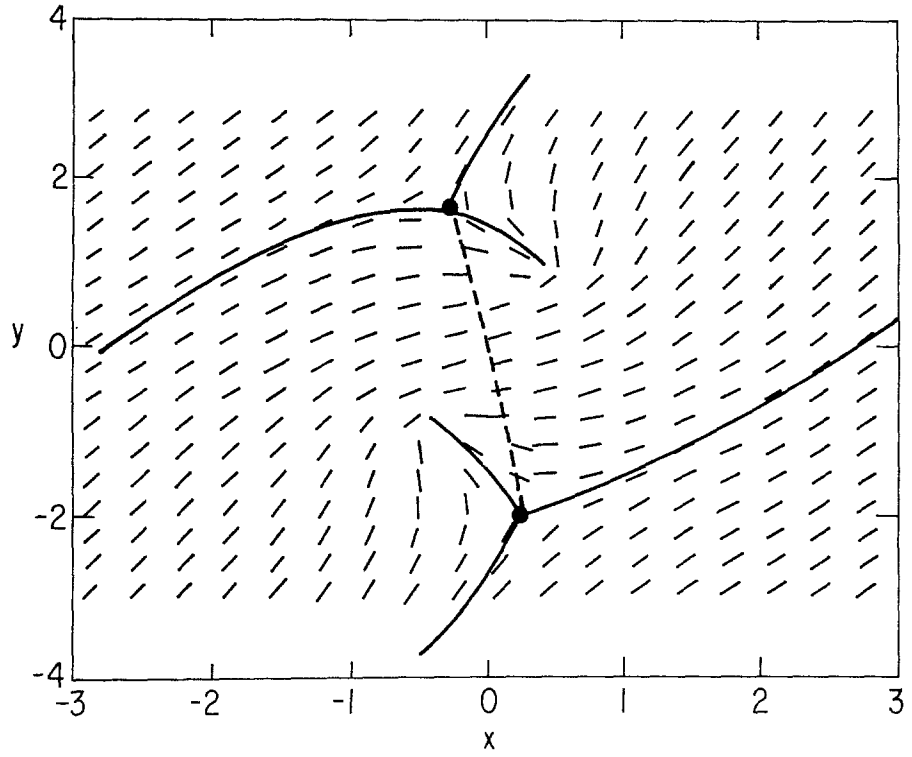

FIG. 5. The Stokes diagram for the initial frequency guess for the equation of Example 1. The dashed line connecting the two-turning points is the integration path.

the two turning points. Similarly, the cut structure of $[Q(z, \omega)]^{1/2}$ must be taken into account in the continuation of the solution from one region to the next.

The plot of the frequency plane search gives an indication of how rapidly the convergence to the correct eigenvalue is proceeding, and when a value has been bracketed it gives some indication of the accuracy of the determination.

B. *Example* 1.

Consider the simple example given by

$$Q = -[(\omega^2/2) + 13]/[z^2 + 0.7(1 - i)] - \omega. \tag{15}$$

This is an ad hoc equation which is illustrative of the type of problem encountered in plasma physics. The turning point locations and indeed the validity of the WKBJ approximation depend on the value of the eigenvalue ω, and it is not a priori apparent whether or not solutions exist of the bound state form, nor whether they can be described by Phase Integral Methods. An initial guess of $\omega = 3 + 7i$ gives the Stokes plot shown in Fig. 5. Two turning points exist, and it appears possible that a solution can be found which exhibits subdominant behavior for $x \to \pm\infty$. Proceeding with

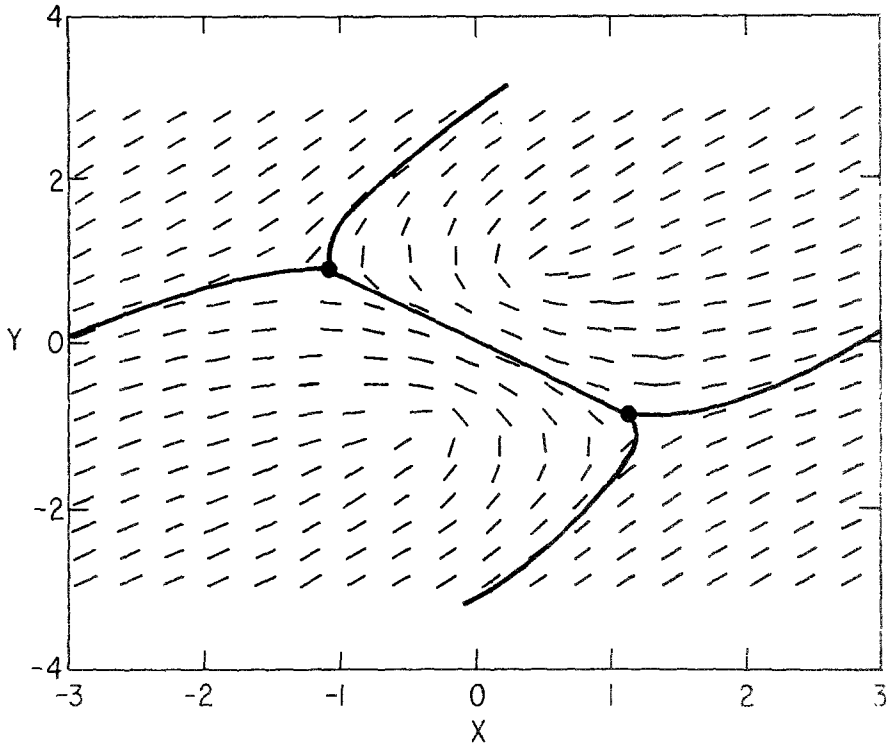

FIG. 6. The Stokes diagram of Example 1 with ω equal to an eigenvalue (that for $n = 2$).

the iteration, we find convergence to an eigenvalue associated with $n = 2$ of $\omega = -1.69 + 8.31i$. The Stokes structure corresponding to this correct eigenvalue is shown in Fig. 6. It is clear that subdominant solutions corresponding to those of regions 1, 6 of Fig. 3 include in their domain of validity the real axis for $x \to \pm\infty$. However, since the Stokes line does not coincide with the real axis, the solution will exhibit propagation. The physical boundary condition which must be imposed is that there be an outward group velocity for large x. We review here the consequences of this for the Stokes analysis. Assume for simplicity that Q has a limit Q_∞ as $x \to \infty$ as is the case in Example 1. This assumption is not essential but simplifies the analysis. Define $k = (Q_\infty)^{1/2}$. The WKBJ solutions for large x are $\psi_\pm \sim \exp(\pm ikx)$. Form a wave packet using ψ_+, $\psi(x, t) = \int dk\, f(k) \exp(ikx - i\omega t)$ where $f(k)$ has a small domain centered about $k = k_0$. Expand ω in a Taylor series about k_0 giving

$$\psi(x, t) = \exp(ik_0x - i\omega_0t) \int dk\, f(k) \exp\{i(k - k_0)[x - (d\omega/dk)\, t]\}. \qquad (16)$$

By standard stationary phase arguments $\psi(x, t)$ is nonzero only in the neighborhood of $x = (d\omega/dk)\, t$ and thus the group velocity is $v_g = d\omega/dk$. Thus ψ_+ is an outgoing wave for $x > 0$ and $v_g > 0$. Now examine the spatial dependence of ψ_+ for large x.

If there are no sources or sinks of energy at large x, Q_∞ is real for real ω, and thus ψ_+ is oscillatory with constant amplitude. If ω is complex, $\omega = \omega_r + i\gamma$, ψ_+ takes the form

$$\psi_+ \approx \exp[ik_0 x - (dk/d\omega)\,\gamma x] \tag{17}$$

and thus a wave with outgoing group velocity is spatially damped for $\gamma > 0$ and spatially growing for $\gamma < 0$. It is readily shown that this property holds for both signs of x and v_g ; i.e., at large x one must choose a subdominant solution for $\gamma > 0$ and a dominant solution for $\gamma < 0$. This behavior has a simple physical interpretation; information regarding the growth or decay of the mode propagates outward with speed v_g and thus the scale length associated with this growth or decay is given by v_g/γ.

The fact that the turning points are complex does not otherwise alter the analysis of the connection formula of Section II. The solution can be regarded as the same, only expressed in a coordinate system which has been rotated and somewhat skewed from that of Fig. 3.

For this particular example the condition for the validity of the WKBJ solution, inequality (3), is fairly well satisfied everywhere on the real axis, as shown in Fig. 7.

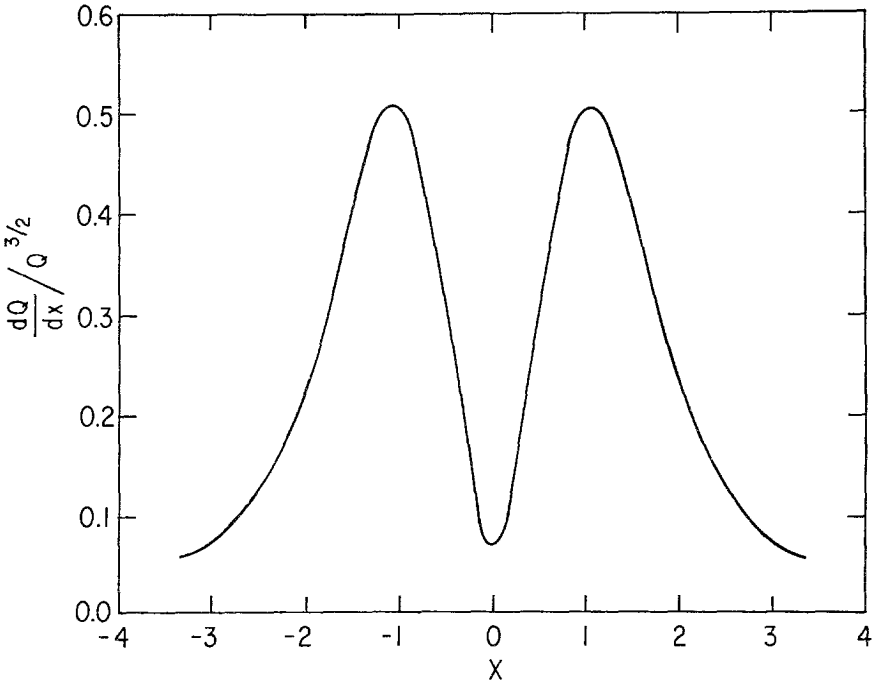

FIG. 7. The *WKBJ* condition $(dQ/dx)Q^{-3/2}$ along the real axis for the case of Fig. 6. This must be small compared to one for the validity of the *WKBJ* solutions.

C. *Example* 2.

Consider the equation defined on the real axis by

$$Q = -\frac{\omega L}{2}\left(K^2 + 1 - \frac{1}{\omega}\right) + \frac{x^2}{4} + \frac{L}{2}(1-\omega)\,AZ(A) \tag{18}$$

where

$$A = (\omega RL)^{1/2}/|\,x\,|$$

and L, K, R are real parameters, and Z is the plasma dispersion function. This equation arises in plasma physics in the consideration of the collisionless drift wave. A more complete description of the derivation and solutions of this equation has been reported elswhere, [7, 8] for our purposes we may simply regard it as providing an example of interesting complexity.

The expression $|\,x\,|$ arises in a derivation valid on the real axis, and the analytic continuation into the complex plane is provided by $|\,x\,| \to (z^2)^{1/2}$. The physical real axis is determined by $|\,x\,| \geqslant 0$, and thus the physical plane is divided into two parts by branch cuts which can be taken along the imaginary axes. Thus for $\mathrm{Re}\ z > 0$, $|\,x\,|$ is replaced by z, and for $\mathrm{Re}\ z < 0$, $|\,x\,|$ is replaced by $-z$. The continuation of these functions through the cuts defines a second plane which we will refer to as the nonphysical plane. It is also divided into two distinct parts. For values of K much less than a critical value K_c which depends on R and L (for $R = 1/1837$ and $L = 50$, $K_c = 0.36$) there are two turning points located approximately along the line $x = -y$ in the physical plane. As K increases, these turning points begin to migrate toward the imaginary axis. In Fig. 8 is shown the location of the turning points (v and $-v$) for a mode with $L = 50$, $K = 0.26$, and $R = 1/1837$. Note the presence of another pair of turning points (g and $-g$), located in the nonphysical plane. Assume a decaying mode, in which case the solution for $x \to \infty$ (Region A of Fig. 8) must be dominant, $\psi(z) = (z, v)_d$. Continuing in toward $x = 0$ (Region B) we cross a Stokes line associated with v (shown as a dotted line in Fig. 8) and thus the solution becomes $\psi(z) = (z, v)_d + i(v, z)_s$. The differential equation and the turning points under consideration are symmetric about $z = 0$, thus we can choose a solution with a particular parity with respect to $z \to -z$. For an odd solution we require $\psi(0) = 0$, or $(0, v) + i(v, 0) = 0$. This has the solution $2\int_0^v Q^{1/2}\,dz = [n + (1/2)]\,\pi$, n odd. For the even solutions we require that $d\psi/dz$ vanish at $z = 0$, or $i(0, v) + (v, 0) = 0$. This has the solution $2\int_0^v Q^{1/2}\,dz = [n + (1/2)]\,\pi$, n even. Thus we obtain the standard connection formula between v, $-v$ which for the parameters given above gives $\omega = 0.908 - 0.008i$. The mode is decaying in agreement with our initial assumption of a purely dominant solution for large x. It is readily verified that the anti-Stokes line emanating from v toward positive x does not cross the real axis.

As the parameter K increases, provided $L > 3R^{-1/4}$, the turning points v, $-v$ as well as the turning points g, $-g$ approach the imaginary axis and coalesce for $K = K_c$,

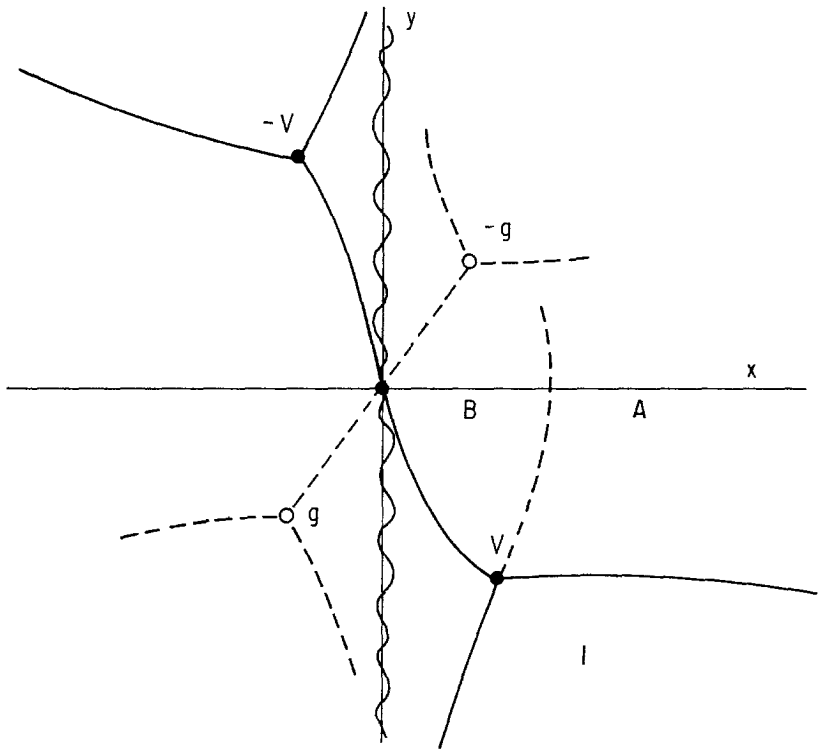

FIG. 8. The Stokes structure for a solution of the equation given in Example 2. Here K^2 is less than the critical value for coalesence of v, g. The turning points g, $-g$ correspond to a mode with positive growth located in the nonphysical plane. Their associated anti-Stokes lines are shown as dashed lines because of their location in the nonphysical plane. The mode in the physical plane associated with v, $-v$ is damped. Also partially shown (dotted line), is the Stokes line extending from v which intersects the real axis.

after which the Stokes diagram takes the form of Fig. 9. The role of turning point for the determination of the solution has been passed on from v to g. Once again the analysis presented above carries through, only now the connection formula is determined by performing the integral from g to $-g$. We then discover that the growth rate $\mathrm{Im}(\omega)$ is zero, which can be verified analytically. If $L < 3R^{-1/4}$ this coalescence does not occur for any value of K, and the mode continues to be determined by the turning point v.

For K less than the critical value; i.e., with Stokes structure as shown in Fig. 8, imposing the connection formula between g and $-g$ gives rise to a growing mode $\mathrm{Im}(\omega) > 0$. However, this ghost mode remains in the nonphysical plane, interpretable as an outwardly propagating solution only in this plane. The critical value of K is seen to be the coalescense of the turning points of a nonphysical growing mode and a physical damped mode. The nonphysical turning point then dominates to produce a marginally stable mode for all larger values of K.

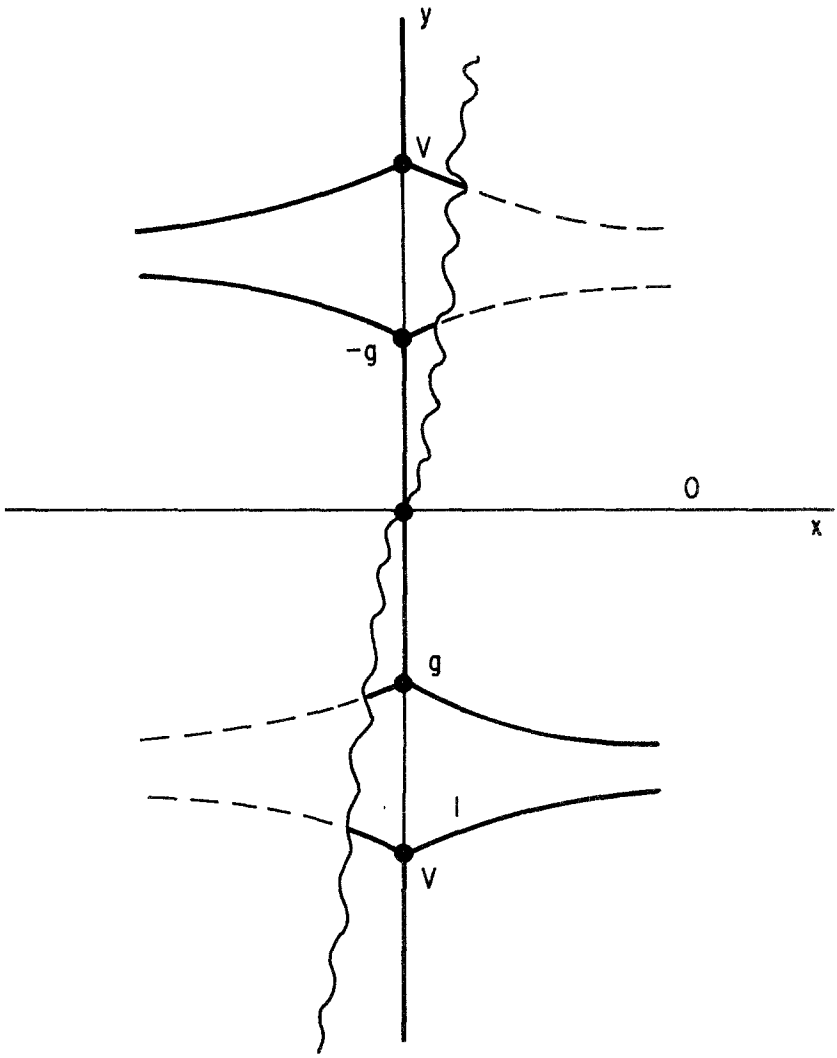

FIG. 9. The Stokes structure for a solution of the equation given in Example 2. Here K^2 is larger than the critical value and the turning points are located on the imaginary axis. The mode is marginally stable.

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